Lecture 17

Last Time: $f : [0, \infty) \to \mathbb{R}$ piecewise continuous and

satisfying: there exists constants α, M and T such that

$$|f(t)| \leq M e^{\alpha t} \quad \text{for all } t \geq T.$$

Translation property: Suppose $a \in \mathbb{R}$. If $s > \alpha$

then $\mathcal{L}\{e^{at} f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a)$.

First Derivatives in t-space: If $f'(t)$ is piecewise continuous

of exponential order α, then

$$\mathcal{L}\{f'(t)\}(s) = s \cdot \mathcal{L}\{f(t)\}(s) - f(0).$$

~~Example:~~ $\cancel{\mathcal{L}\{e^{at}\cos(\omega t)\} =}$

Example: Given that $\mathcal{L}\{\sin(\omega t)\}(s) = \frac{\omega}{\omega^2+s^2}$ with $s > 0$.

Calculate $\mathcal{L}\{e^{at}\cos(\omega t)\}(s)$.

We have $\mathcal{L}\{e^{at}\cos(\omega t)\}(s)$

$$= \mathcal{L}\{\cos(\omega t)\}(s-a) \quad , \quad s > a.$$

$$= \frac{1}{\omega}\mathcal{L}\{\sin'(\omega t)\}(s-a)$$

$$= \frac{1}{\omega}\left((s-a)\mathcal{L}\{\sin(\omega t)\}(s-a) - \sin(\omega \cdot 0)\right)$$

$$= \frac{s-a}{\omega^2+(s-a)^2} \quad , \quad s > a.$$

Derivatives in s-space.

Goal: Understand how the operation $f(t) \longmapsto t^n f(t)$ changes when we apply $\mathcal{L}$.

Theorem: Suppose $f(t)$ is piecewise continuous of exponential order α.

If n is a positive integer and $s > \alpha$

then $\mathcal{L}\{t^n \cdot f(t)\}(s) = (-1)^n \cdot \frac{d^n}{ds^n}\left(\mathcal{L}\{f(t)\}(s)\right)$.

Proof Sketch: For each $n \in \mathbb{Z}_{\geq 0}$ define

$$f_n : [0, \infty) \longrightarrow \mathbb{R}, \quad f_n(t) = t^n \cdot f(t).$$

~~Then~~ $\cancel{\mathcal{L}\{f_0(t)\}(s) =}$

$$\cancel{= (-1)^0 \cdot \frac{d^0}{ds^0}\left(\mathcal{L}\{f(t)\}(s)\right)}.$$

~~So the theorem holds for~~ ________.

For general $n \geq 1$ we have

$$\mathcal{L}\{f_n(t)\}(s) = \mathcal{L}\{t \cdot f_{n-1}(t)\}(s)$$

$$= \int_0^\infty e^{-st} \cdot t \cdot f_{n-1}(t)\, dt$$

$$= \int_0^\infty -\frac{d}{ds}\left(e^{-st}\right) \cdot f_{n-1}(t)\, dt$$

$$= -\frac{d}{ds}\left(\int_0^\infty e^{-st} \cdot f_{n-1}(t)\, dt \right)$$

where the last equality follows from the <u>Liebnitz's</u> <u>Rule</u>.

So $\mathcal{L}\{f_n(t)\}(s) = -\frac{d}{ds}\left(\mathcal{L}\{f_{n-1}(t)\}(s)\right)$

$$= -\frac{d}{ds}\left(-\frac{d}{ds}\left(\mathcal{L}\{f_{n-2}(t)\}(s)\right)\right)$$

$$= \frac{d^2}{ds^2}\left(\mathcal{L}\{f_{n-2}(t)\}(s)\right)$$

$$= -\frac{d^3}{ds^3}\left(\mathcal{L}\{f_{n-3}(t)\}(s)\right)$$

$$\mathcal{L}\{f_n(t)\}(s) = \dots = (-1)^n \cdot \frac{d^n}{ds^n}\left(\mathcal{L}\{f_0(t)\}(s)\right)$$

Since $f_0(t) = t^0 \cdot f(t)$, We conclude

$$\mathcal{L}\{t^n \cdot f(t)\}(s) = (-1)^n \cdot \frac{d^n}{ds^n}\left(\mathcal{L}\{f(t)\}(s)\right).$$

Example: Calculate $\mathcal{L}\{t^2 \sin(\omega t)\}$.

Applying the previous theorem we have

$$\mathcal{L}\{t^2 \sin(\omega t)\}(s)$$

$$= -\frac{d}{ds}\left(\mathcal{L}\{t \sin(\omega t)\}(s)\right)$$

$$= \frac{d^2}{ds^2}\left(\mathcal{L}\{\sin(\omega t)\}(s)\right).$$

So $\mathcal{L}\{t^2\sin(\omega t)\}(s) = \frac{d^2}{ds^2}\left(\frac{\omega}{s^2+\omega^2}\right)$

Simplifying we obtain

$$\mathcal{L}\{t^2\sin(\omega t)\}(s) = \frac{d}{ds}\left(\frac{-2s\omega}{(s^2+\omega^2)^2}\right)$$

$$= -2\omega \cdot \frac{(s^2+\omega^2)^2 - s\cdot 2(s^2+\omega^2)\cdot 2s}{(s^2+\omega^2)^4}$$

$$= \frac{2\omega(3s^2-\omega^2)}{(s^2+\omega^2)^3}, \quad s>0.$$

Inverse Laplace Transform.

<u>Question</u>: Can we <u>undo</u> the operation of taking Laplace transform?

Warm up: Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function. If it exists the inverse of f is the unique function f^{-1} satisfying: if $x \in \mathbb{R}$ and

$$y = f(x)$$

then $f^{-1}(y) = x$.

Remark: (i) f^{-1} can fail to exist.

(ii) If f^{-1} exists then f^{-1} is unique.

Definition: Suppose $F(s)$ is a function in a variable s. If it exists, the inverse Laplace transform is the unique continuous function

$$\mathcal{L}^{-1}\{F\} : [0, \infty) \longrightarrow \mathbb{R}$$

such that if $f(t) = \mathcal{L}^{-1}\{F\}(t)$

then $\mathcal{L}\{f(t)\}(s) = F(s)$.

Examples: Determine whether the following functions have inverse Laplace transforms.

(a) $F(s) = 2/s^3$, $s > 0$.

From our table of Laplace transforms we know that for $n \in \mathbb{Z}_{\geq 1}$ we have

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}, \quad s > 0.$$

Hence $f(t) = \underline{t^2}$ is a <u>continuous</u> function on $[0, \infty)$ satisfying $\mathcal{L}\{f\} = F$.

~~So~~ $\cancel{\mathcal{L}^{-1}\{2/s^3\}(t) = }$

So $\mathcal{L}^{-1}\{2/s^3\}(t) = \underline{t^2}$.

(b) $$F(s) = \frac{s-1}{s^2 - 2s + 5}, \quad s > 1.$$

Completing the square we have

$$s^2 - 2s + 5 = \underbrace{s^2 - 2s + 1} + 4$$

$$= (s-1)^2 + 4.$$

So $F(s) = \dfrac{2}{(s-1)^2+4}$.

Consulting our table of Laplace transforms we note that $F(s) = \mathcal{L}\{\sin(2t)\}(s-1)$
$= \mathcal{L}\{e^t \sin(2t)\}(s)$.

Hence by the translation property

$$\mathcal{L}^{-1}\{F(s)\}(t) = e^t \sin(2t).$$

(c) $F(s) = \dfrac{3s+2}{s^2+2s+10}$, $s > 1$.

By completing the square we have

$$s^2+2s+10 = s^2+2s+1+9$$
$$= (s+1)^2+9.$$

Hence $F(s) = \dfrac{3s+2}{(s+1)^2+9}$.

Consulting our table of Laplace transform we note $\mathcal{L}\{e^{-t}\sin(3t)\} = \dfrac{3}{(s+1)^2+9}$

and $\mathcal{L}\{e^{-t}\cos(3t)\} = \dfrac{s+1}{(s+1)^2+9}$.

Hence we write

$$(*) \qquad F(s) = A \cdot \frac{s+1}{(s+1)^2+9} + B \cdot \frac{3}{(s+1)^2+9}$$

where A and B are constants.

Equating the numerators in $(*)$ gives

$$3s+2 = \underline{A(s+1) + B \cdot 3}.$$

By equating coefficients on the powers of s we obtain $A = 3$ and $A + 3B = 2$

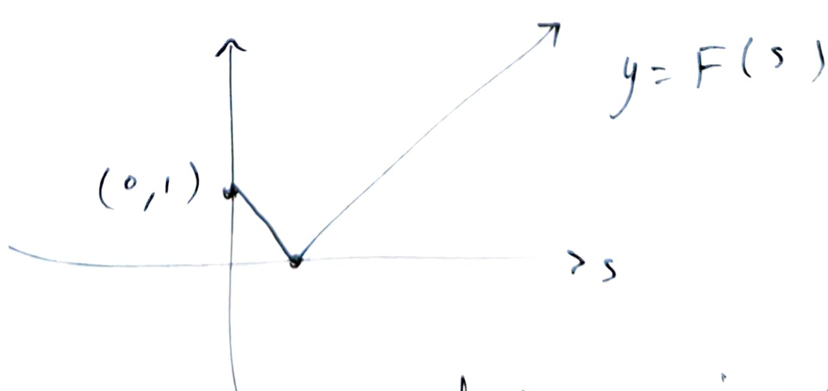
$$\Rightarrow B = -1/3.$$

So
$$F(s) = 3 \cdot \frac{s+1}{(s+1)^2 + 9} - \frac{1}{3} \cdot \frac{3}{(s+1)^2 + 9}$$

$$= 3 \cdot \mathcal{L}\{e^{-t}\cos(3t)\}(s) - \frac{1}{3}\mathcal{L}\{e^{-t}\sin(3t)\}(s)$$

Hence $\mathcal{L}^{-1}\{F(s)\}(t) = \underline{3e^{-t}\cos(3t) - \frac{1}{3}e^{-t}\sin(3t)}$.

<u>Example</u>: $F(s) = |s - 1|$, $s > 0$.



<u>claim</u>: $F(s)$ does not admit an inverse Laplace transform

<u>Recall</u>: If f is piecewise continuous and $\mathcal{L}\{f(t)\}(s)$ exists for $s > 0$ then $\mathcal{L}\{t^n \cdot f(t)\}(s) = (-1)^n \cdot \frac{d^n}{ds^n} \mathcal{L}\{f(t)\}(s)$.

In particular $\mathcal{L}\{f(t)\}(s)$ has derivatives of all orders on $(0, \infty)$.

However $F(s) = |s-1|$ does not have a continuous at $s=1$. So $\frac{d^2}{ds^2} F(s)$ is undefined at $s=1$. So $F(s) = \mathcal{L}\{f(t)\}(s)$

$$\rightarrow F''(1) = \mathcal{L}\{t^2 f(t)\}(1).$$

Contradiction

$\Rightarrow F(s)$ does not admit an inverse Laplace transform.